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LETTER TO THE EDITOR

On the integrability of Schamel's modified Korteweg–de Vries equation

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Abstract. A recent suggestion of integrability of Schamel's modified Korteweg–de Vries equation for ion acoustic waves in a two-component plasma is discussed. Various arguments, including ones based on the truncated Painlevé expansion, are presented which support the existence of special structure for this equation. However, the evidence for complete integrability of Schamel's equation is not yet conclusive.

Recently a form of the modified Korteweg–de Vries (mKdV) equation with three halves degree of nonlinearity has received attention [1] from the point of view of Painlevé analysis (e.g., [2–6]. In [1] Xiao obtained results with the use of the reduced or Kruskal ansatz [4,6] for the singularity manifold $\phi(x,t) = x + \psi(t)$ and expansion coefficients $V_j(x,t) = V_j(t)$, $j = 0,1,\ldots,\infty$. For the case of Schamel's mKdV equation [7–9]

$$16u_t + u_{xxx} + 30u^{1/2}u_x = 0 \tag{1}$$

Xiao investigated the equation

$$16q^2q_t + 6q_x^3 + 9qq_xq_{xx} + q^2q_{xxx} + 30q^4q_x = 0 \tag{2}$$

which results from the change of variable $u = q^4$, showing that compatible resonances occur at $j = -1, 6$, and 10 [10,26]. In this letter we present a simpler nonlinear transformation with which to study generalized mKdV equations of the form considered by Xiao, using it as a starting point for a Painlevé analysis, and discuss several questions posed by his analysis. We also mention the Hamiltonian formalism for Schamel's equation and present a limited number of corresponding conservation laws. We find aspects of special structure including a candidate Miura map, Bäcklund transformation, and a Lax pair which may bear further investigation. However, contrary to the implied conclusion of [1], the evidence for complete integrability of Schamel's equation is not yet conclusive.

Equation (1) arises in plasma physics in the study of ion acoustic solitons when electron trapping is present (e.g. [9,11]). Equation (1) governs the electrostatic potential for a certain electron distribution in velocity space [9]. Schamel's equation also arises in condensed matter physics in the study of the thermal conductivity of certain (model) solids. Equation (1) occurs in describing phonons for a one-dimensional nonlinear lattice when the interparticle force includes a term proportional to the three-halves power of the displacement [12]. In this setting, Schamel's equation follows for a derivative of the displacement in a continuum approximation with a semicharacteristic variable-stretching transformation [12]. As pointed out by Schamel [7, 8], (1) possesses a stronger nonlinearity than the usual KdV equation in that the single soliton solution has a smaller width and higher velocity. An explicit pulse-type (sech^4) soliton solution of Schamel's equation has been found by several authors [7–9, 13–14]. In addition, a generalized KdV equation for which (1) is a special case has been studied in a variety of mathematical physics contexts. A few references in this regard are [14, 16] for general properties and [12, 15, 17, 18] for particular solutions and physical applications. Related KdV-like equations and their physical origins are discussed in [19, 20]. (None of these instances of Schamel's equation or generalized KdV equation are mentioned in [1].)

We first make some observations concerning the Painlevé analysis performed by Xiao [1] for the equation

$$n^2 u_t + u_{xxx} + (n+1)(n+2)u^{2/n}u_x = 0 \tag{3}$$

which by taking $u = q^n$ becomes

$$n^2q^2q_t + (n-1)(n-2)q_x^3 + 3(n-1)qq_xq_{xx} + q^2q_{xxx} + (n+1)(n+2)q^4q_x = 0. \tag{4}$$

Equation (3) can also be written in potential form with $\theta = u_x$,

$$n^2\theta_t + \theta_{xxx} + n(n+1)\theta_x^{2/n+1} = c \tag{5}$$

where c is a constant of integration. A guarantee of the non-negativity of the solution of (3) for even values of n was not given in [1]. In the following we assume that such non-negative solutions exist under not too restrictive conditions on the initial data and boundary conditions.

Equation (4) has polynomial nonlinearity of degree three except for the last term of degree five. (The case $n = 4$ is Schamel's equation.) The importance of the truncated Painlevé expansion in yielding information such as Bäcklund transformations, Miura maps, Lax pairs, Hirota's bilinear representation, and special and rational solutions is well-known (e.g., [2–5, 22]). The singularity exponent α being -1 for (4), the Painlevé expansion truncated at the constant level becomes simply $q_T(x,t) = V_0(x,t)/\phi(x,t) + V_1(x,t)$. (The T subscript on q shall denote the truncated Painlevé series throughout this letter.) In the case of the reduced ansatz, $V_0 = \pm\mathrm{i}\phi_x = \pm\mathrm{i}$ and $V_1 = 0$ [1]. Although V_1 is a (trivial) solution of equation (4), the form $q_T = \mathrm{i}/\phi$ with $\phi_x = 1$ is insufficient to bilinearize (4), leading only to the relation $\phi_t = 0$. The solution thus found (in the context of the reduced ansatz) is $u(x) = (\pm\mathrm{i})^n/(x+x_0)^n$, where x_0 is a constant, a special case of the solution obtained from (3) by direct integration with $u_t = 0$. Now for a truncated expansion, the reduced form of the singularity manifold is usually *not* assumed [4]. Therefore it is of interest to remove the restriction that $\phi_x = 1$. As verified by direct calculation, the use of $q_T(x,t) = \mathrm{i}(\mathrm{d}/\mathrm{d}x)\ln\phi(x,t) = \mathrm{i}\phi_x/\phi$ results in a fourth-order equation for ϕ each term of which has a sixth degree nonlinearity:

$$n^2\phi^2\phi_x^2(\phi\phi_{xt}+\phi_x\phi_t)+(n-1)(n-2)\phi^3\phi_{xx}^3+3(n-1)\phi^3\phi_x\phi_{xx}\phi_{xxx}+\phi^3\phi_x^2\phi_{xxxx} \\ +(2n+1)(n+1)\phi\phi_{xx}\phi_x^4-3n^2\phi^2\phi_x^2\phi_{xx}^2-(3n+1)\phi^2\phi_x^3\phi_{xxx}=0. \tag{6}$$

Although simple exponential functions solve equation (6), they are of little interest, as they result in a trivial q_{T}. However, in light of the known single solitary wave solution for general n (e.g., [14, 15, 17]), presumably useful hyperbolic trigonometric solution(s) exist of this sixth-degree equation for the singularity manifold function ϕ. In a sense, (6) represents a homogenization of the nonlinearity of (4). We stress, however, that the mathematical cost of this transformation is to increase the degree of nonlinearity rather than to decrease it, as would be expected in the case of a completely integrable equation. In particular, for $n = 4$, (6) still has sixth degree of nonlinearity. Only if (6) can be suitably integrated will a bilinear equation follow, as expected if (2) were completely integrable like the KdV equation.

Successful passage of the Painlevé test as described by Weiss is considered a sufficient condition for complete integrability [3] (p 200). However, the nature of the singularity manifold itself needs to be examined. According to Weiss [3] (p 178) all movable singularity manifolds must be single-valued, whether characteristic or not.

Now the transformation of (3) to (4) is unnecessarily complicated. The change of variable $u = q^{n/2}$ [15] in (3) gives

$$n^2q^2q_t+\left(\tfrac{1}{2}n-1\right)\left(\tfrac{1}{2}n-2\right)q_x^3+3\left(\tfrac{1}{2}n-1\right)qq_xq_{xx} \\ +q^2q_{xxx}+(n+1)(n+2)q^3q_x=0. \tag{7}$$

(This equation then has an explicit sech^2 solution.) An obvious advantage of (7) over (4) for general n is that (7) has quartic versus quintic highest degree nonlinearity. Furthermore, for $n = 4$ (the Schamel equation) the q_x^3 term in (7) is absent and a factor of q can be cancelled throughout. The resulting equation,

$$16qq_t+3q_xq_{xx}+qq_{xxx}+30q^2q_x=0 \qquad (n=4) \tag{8}$$

has only cubic highest degree nonlinearity.

The leading order singularity of (8) is found to be $\alpha \doteq -2$, as expected on the basis of Xiao's result. Developing a Painlevé expansion in the form $q(x,t) = \phi^{-2}(x,t)\sum_{j=0}^{\infty}q_j(x,t)\phi^j(x,t)$ shows that the resonances are unchanged, being at $j = 6$ and $j = 10$. The general nonlinear recursion relation (RR) for the expansion coefficients q_j is a little too lengthy to give here. It gives $q_0 = -\phi_x^2$ (a real function in distinction to V_0), $q_1 = \phi_{xx}$,

$$q_2=-(4\phi_t-3\phi_{xx}^2/\phi_x+4\phi_{xxx})/12\phi_x \tag{9a}$$

and with the help of a MACSYMA program [21]

$$12\phi_x^4q_3=\phi_x^2\phi_{xxxx}-4\phi_x\phi_{xx}\phi_{xxx}+3\phi_{xx}^3-4\phi_t\phi_x\phi_{xx}+4\phi_{tx}\phi_x^2. \tag{9b}$$

Proceeding further with the use of the reduced ansatz $\phi_x = 1,\ \phi_t = \psi_t,\ q_j = q_j(t)$ we have the RR

$$\left\{16\left[\sum_{j=3}^{\infty}\sum_{k=0}^{j-3} q_{j-k-3}q_{k,t} + \sum_{j=2}^{\infty}\sum_{k=0}^{j-2}(k-2)q_{j-k-2}q_k\psi_t\right] + 30\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{k}(\ell-2)q_{j-k}q_{k-\ell}q_\ell + \sum_{j=0}^{\infty}\sum_{k=0}^{j}(k-2)(k-3)(3j-2k-10)q_{j-k}q_k\right\}\phi^{j-7} = 0 \tag{10}$$

which gives [21]

$$q_0 = -1 \qquad q_1 = 0 \qquad q_2 = -\psi_t/3 \qquad q_3 = 0 \tag{11a}$$
$$q_4 = -\psi_t^2/15 \qquad q_5 = 8\psi_{tt}/45 \qquad q_7 = -8\psi_t\psi_{tt}/45 \tag{11b}$$
$$q_8 = [160\psi_{ttt} - 3\psi_t^4]/2025 \qquad q_9 = 8[-8\psi_t^2\psi_{tt} + 45q_{6t}]/675. \tag{11c}$$

The resonances are compatible, i.e., the functions q_6 and q_{10} are arbitrary in the Painlevé expansion. Therefore (8) passes the Painlevé test in the strong form: all resonances are at integer values and are compatible—in which sense the Schamel equation is partially integrable. However, the Painlevé expansion truncated at the constant level,

$$q_{\mathrm{T}}(x,t) = -\frac{\phi_x^2}{\phi^2} + \frac{\phi_{xx}}{\phi} + q_2(x,t) = \frac{\partial^2}{\partial x^2}\ln\phi + q_2(x,t) \tag{12}$$

with the function $q_2 = 0$ fails to bilinearize (8). An immediate difficulty can be seen in attempting to integrate the term with the time derivative with respect to x once. The transformation (12) for (8) is very like that for the KdV equation (e.g., [2,3], [22] p 172). Without a means of integrating the resulting equation, it is not clear how Hirota's bilinear representation can be obtained in the usual way.

Nonetheless, proceeding as for the KdV equation (3), we can find a singularity manifold equation which is bilinear in ϕ. This then leads us to a possible linearization of (8). By imposing the condition $q_3 = 0$ we have the equation

$$\phi_x^3\frac{\partial}{\partial x}\left[4\frac{\phi_t}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 + \frac{\phi_{xxx}}{\phi_x}\right] = 0 \tag{13}$$

which can be integrated once to give

$$4\phi_x\phi_t - \tfrac{3}{2}\phi_{xx}^2 + \phi_x\phi_{xxx} = \lambda\phi_x^2 \tag{14}$$

where λ is a constant. Using the Schwarzian derivative $\{\phi; x\}$, the unique differential invariant of the Möbius group (e.g., [3]), (14) can be written as

$$\lambda = 4\frac{\phi_t}{\phi_x} + \{\phi; x\}. \tag{15}$$

We further have the relations

$$q_2 + \frac{1}{12}\lambda = -\frac{1}{4}\frac{\phi_{xxx}}{\phi_x} + \frac{1}{8}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \qquad (16a)$$

$$q_2 + \frac{1}{12}\lambda = -\frac{1}{4}\frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{8}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \qquad (16b)$$

where the latter can be viewed as a candidate Miura transformation. Equation (16*b*) is a Ricatti equation in the variable $V \equiv \phi_{xx}/\phi_x$, linearizable by the substitution $V = 2v_x/v$. This gives the linear Schrödinger equation

$$-\tfrac{1}{2}v_{xx} - q_2 v = \tfrac{1}{12}\lambda v \qquad (17)$$

which is part of a candidate Lax pair for (8). Here, $v^2 = \phi_x$ plays the role of a squared eigenfunction with corresponding eigenvalue $\lambda/12$ and q_2 is the potential in the associated scattering problem. The other part of the Lax pair is given by

$$v_t = \left(\tfrac{1}{6}\lambda - q_2\right) v_x + \tfrac{1}{2}q_{2,x}v. \qquad (18)$$

Equation (18) can be obtained by: solving for λ from (16a) and setting the result equal to expression (15), differentiating with respect to x, and eliminating ϕ_t with the aid of (15). We have thus found a candidate Lax pair and Bäcklund transformation, (16), for (8).

Using the compatibility condition $v_{xxt} = v_{txx}$, the candidate Lax pair (17), (18) provides the KdV equation for $q_2(x,t)$: $4q_{2,t} + q_{2,xxx} + 12q_2q_{2,x} = 0$. By using the singularity manifold equation (14), it is not difficult to show that the function V satisfies the mKdV equation $4V_t + V_{xxx} - \frac{3}{2}V^2V_x = \lambda V_x$.

The linearizing substitution for ϕ is expected to be $\phi = v_1/v_2$ where v_1, v_2 solve the Schrödinger equation (17) (e.g., [3]). Yet these results are insufficient to show that (8) has N soliton solutions, in which sense Schamel's equation could be considered completely integrable. Among the items to be examined is the consistency of setting the higher-level q_j's to zero, especially the conditions $q_4 = q_5 = 0$ and likewise for q_7 through q_9.

The Schamel equation has at least three conservation laws and the candidate Lax pair opens the possibility of finding an infinite number, characteristic of completely integrable equations. One of these laws is equation (1) itself, written in the form $16u_t + F_{1x} = 0$ where the flux $F_1 = u_{xx} + 20u^{3/2}$. Another conservation law is $(8u^2)_t + F_{2x} = 0$ where the flux $F_2 = -u_x^2/2 + uu_{xx} + 12u^{5/2}$. (These two conservation laws obviously extend for arbitrary n for the generalized KdV equation (3).) The first of these laws may be termed conservation of a momentum and the second conservation of an energy.

Here we can only touch on the subject of the Hamiltonian structure for Schamel's equation. It may be pointed out that non-unique Lagrangian and Hamiltonian densities for (1) exist. One possible Lagrangian density [17, 19] for (1) is

$$\mathcal{L} = 8\theta_x\theta_t - \theta_{xx}^2/2 + 8\theta_x^{5/2} \qquad (19)$$

where θ is a potential function, $\theta_x = u$. The corresponding Hamiltonian density is

$$\mathcal{H} = \theta_t\Pi - \mathcal{L} = u_x^2/2 - 8u^{5/2} \qquad (20)$$

where the canonical momentum density is $\Pi \equiv \partial\mathcal{L}/\partial\theta_t = 8\theta_x$. The latter relation enables us to state that $P = \int \Pi\, \mathrm{d}x = 8\int_{-\infty}^{\infty} u\, \mathrm{d}x$ is a total conserved momentum. The Hamiltonian is given by $H = \int_{-\infty}^{\infty} \mathcal{H}\, \mathrm{d}x$. By using the equation of motion (1), integrating by parts, and assuming that the first two spatial derivatives of u vanish at infinity, we find that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{-\infty}^{\infty} (u_x u_{xt} - 20u^{3/2}u_t)\, \mathrm{d}x = \frac{25}{2}u^3\bigg|_{-\infty}^{\infty} = 0 \tag{21}$$

for solutions like the single soliton that either vanish or approach the same constant value at $\pm\infty$. Therefore the Hamiltonian H provides a third constant of the motion, with the corresponding conservation law being $\mathcal{H}_t + F_{3x} = 0$, where the flux

$$F_3 = \tfrac{1}{16}\left[-\tfrac{1}{2}u_{xx}^2 + u_x u_{xxx} - 20u^{3/2}u_{xx} + 30u^{1/2}u_x^2 - 200u^3\right]. \tag{22}$$

Like $H_2 \equiv 8\int_{-\infty}^{\infty} u^2 \mathrm{d}x$, H represents a kind of total conserved energy.

The three conservation laws that we have presented are direct consequences of Noether's theorem [23]. In particular, they follow from the invariance of the Lagrangian density with respect to translations of x, t, or θ. The quantities P and H are the generators of space and time translation and the coordinate θ is cyclic [23]. The further application [24] of Lagrangian and variational techniques (using the action) and similarity reduction for Schamel's equation will not be presented here.

For equations like KdV which have a Lax pair there exist two distinct Hamiltonian structures [25]. For Schamel's equation, the three densities Π, $8u^2$, and $\mathcal{H}$ provide candidate Hamiltonians, but it is not clear if these lead to an infinite ladder structure [25]. A closely related topic is the existence of a Miura-type transformation for Schamel's equation to or from an equation possessing a Hamiltonian structure [25].

It is textbook knowledge [22] (p 263) that the generalized KdV equation $u_t + u_{xxx} + f(u)u_x = 0$ has a scalar pseudopotential only when the function f is quadratic in u, which covers the known completely integrable KdV and mKdV cases. This result assumes that the pseudopotential depends on u and its spatial derivatives up to the second-order, which is usual for an equation of third-order in the space coordinate. The existence of a pseudopotential for Schamel's equation remains an open question.

In summary, the Schamel equation (1) has some special structure associated with it, e.g., the existence of two arbitrary functions in a Painlevé expansion, a possible Miura map, and Lax pair. These results provide evidence that Schamel's equation may be completely integrable. Currently it is not known for Schamel's equation if there exists an infinite number of conservation laws and a Hirota bilinear representation and associated multi-soliton solutions. The truncated Painlevé expansion yields a Darboux transformation in the usual form. The appearance of the Schrödinger equation (17) suggests that inverse scattering theory [22] may be applicable in some form.

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Note added in proof. After the present paper was submitted an article by Ramani and Grammaticos [26] appeared which comments on the Painlevé analysis of Schamel's equation.

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[10] The following (inexhaustive) typographic errors and omission in [1] may be noted. In (4) q_x^3 should replace q^3, equation (5) was substituted into equation (4) not (2), and the Painlevé condition at $j = 9$, which reads $V_9 = 8(8V_0\psi_t^2\psi_{tt} + 45V_{6t})/675$, was not listed.

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